THE SCHWARZIAN DERIVATIVE

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If we identify $\mathbb{C}\mathrm{P}^1$ with the Riemann sphere, then in a coordinate z, the action of $\mathrm{SL}_2\mathbb{C}$ on $\mathbb{C}\mathrm{P}^1$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

But notice that $M \cdot z = (-M) \cdot z$. So to get a faithful action we quotient by the normal subgroup $\{\pm Id\}$, obtaining the group $\mathrm{PSL}_2\mathbb{C} = \mathrm{SL}_2\mathbb{C}/\{\pm Id\}$.

Lemma. *Suppose Ω is an open connected subset of $\mathbb{C}\mathrm{P}^1$ and $f : \Omega \to \mathbb{C}\mathrm{P}^1$ is a locally injective holomorphic function. Given $z \in \Omega$, there exists a unique Möbius transformation $M_f(z) \in \mathrm{PSL}_2\mathbb{C}$ that agrees with f at z to 2nd order, i.e.,*

$$f(w) = M_f(z) \cdot w + o(w-z)^2.$$

Another way to say this is that for fixed $z \in \Omega$,

$$M_f(z) \cdot z = f(z),$$
$$\frac{d}{dw}\left(M_f(z) \cdot w\right)\Big|_{w=z} = f'(z),$$
$$\frac{d^2}{dw^2}\left(M_f(z) \cdot w\right)\Big|_{w=z} = f''(z).$$

The assignment $z \mapsto M_f(z)$ defines a map $M_f : \Omega \to \mathrm{PSL}_2\mathbb{C}$ that is called the Osculating Möbius Transformation of f. If neither z nor $f(z)$ is infinity, then the osculating Möbius transformation is given by

$$M_f(z) \cdot w = \frac{(f'(z)^2 - \frac{1}{2}f(z)f''(z))(w-z) + f(z)f'(z)}{-\frac{1}{2}f''(z)(w-z) + f'(z)}$$

so that

$$M_f(z) = \frac{1}{f'(z)^{3/2}} \begin{pmatrix} f'(z)^2 - \frac{1}{2}f(z)f''(z) & -(f'(z)^2 - \frac{1}{2}f(z)f''(z))z + f(z)f'(z) \\ -\frac{1}{2}f''(z) & f'(z) + \frac{1}{2}zf''(z) \end{pmatrix}.$$

Note that the ambiguity of the $f'(z)^{3/2}$ is taken care of by the quotient to $\mathrm{PSL}_2\mathbb{C}$.

If f is already a Möbius transformation then $M_f(z) = f$ for all $z \in \Omega$. Indeed, $M_f : \Omega \to \mathrm{PSL}_2\mathbb{C}$ is constant if and only if f is a Möbius transformation. Therefore, the derivative of the osculating Möbius transformation should give some measure of how far the function f is from being a Möbius transformation.

The differential $dM_f : T\Omega \to T\mathrm{PSL}_2\mathbb{C}$ takes values in the tangent bundle of $\mathrm{PSL}_2\mathbb{C}$. The tangent bundle of a Lie group is canonically trivialized by left translation:

$$T\mathrm{PSL}_2\mathbb{C} \simeq \mathrm{PSL}_2\mathbb{C} \times \mathrm{Lie}(\mathrm{PSL}_2\mathbb{C}) = \mathrm{PSL}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$$
$$(M, v) \mapsto (M, d(L_{M^{-1}})_M(v)) = (M, M^{-1}v).$$

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So composing dM_f with the projection to $\mathfrak{sl}_2\mathbb{C}$ we can consider the Darboux derivative of f: a 1-form on Ω with values in $\mathfrak{sl}_2\mathbb{C}$. See [Sha97, Chapter 3] for a discussion of Darboux derivatives. An explicit computation gives

$$M_f(z)^{-1}d(M_f)_z = \frac{1}{2}\left(\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right)\begin{pmatrix} -z & z^2 \\ -1 & z \end{pmatrix} dz.$$

So we see that Darboux derivative is zero precisely when the quantity

$$S(f)(z) = \left(\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right) = 0,$$

and so f is a Möbius transformation precisely when $S(f) = 0$. We call $S(f)$ the Schwarzian derivative of f.

The Schwarzian derivative has an interesting chain rule: if f and g are locally injective and holomorphic then a computation shows

$$S(f \circ g) = (S(f) \circ g)(g')^2 + S(g).$$

This suggests that the Schwarzian is more naturally a quadratic differential; that we should redefine

$$S(f)(z) = \left(\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right) dz^2$$

so that the chain rule can be written more cleanly as

$$S(f \circ g) = g^*S(f) + S(g).$$

However, this new definition is not well behaved under a change of coordinates. Say we have a Riemann surface X and a holomorphic map $f : X \to \mathbb{C}$. We could try to define the Schwarzian of f in charts and pull it back to X. That is, suppose $z : U \to \mathbb{C}$ is a coordinate chart, we could try to define $S(f)$ on U by $z^*S(f \circ z^{-1})$. To check if this can be globally defined, take another chart w overlapping with z. Then we have

$$\begin{aligned} z^*S(f \circ z^{-1}) &= z^*S(f \circ w^{-1} \circ (w \circ z^{-1})) \\ &= z^*((w \circ z^{-1})^*S(f \circ w^{-1}) + S(w \circ z^{-1})) \\ &= w^*S(f \circ w^{-1}) + z^*S(w \circ z^{-1}). \end{aligned}$$

So we see we can only patch together the Schwarzian if $S(w \circ z^{-1}) = 0$ for all holomorphic charts on X. That is, only when all the transition functions are Möbius transformations. This leads us to the definition of a Complex Projective Structure.

Definition. *Let S be a smooth surface. A complex projective structure Z on S is an atlas of charts to $\mathbb{CP}^1$ such that all the transition functions are (the restrictions of) Möbius transformations. We refer to S with Z as a complex projective surface.*

Notice that since Möbius transformations are holomorphic, a complex projective structure Z induces a complex structure X on S. Like in the smooth manifolds case, we call a function between two complex projective surfaces $f : Z \to W$ projective if it is (the restriction of) Möbius transformations in all projective charts of Z and W. Again, since Möbius transformations are holomorphic, projective maps between complex projective surfaces are holomorphic with respect to the underlying complex structures.

Suppose $f : Z \to W$. We can define the Schwarzian derivative of f as a quadratic differential on Z. If f is locally injective, then $f'(z) \neq 0$ for any z and so the Schwarzian of f is holomorphic with respect to Z's underlying complex structure. If f is not locally injective, then we have $f'(z) = 0$ somewhere, in which case the Schwarzian is a meromorphic quadratic differential. Now, to define the Schwarzian, take projective charts z for Z and w for W and locally define it by $z^* S(w \circ f \circ z^{-1})$. Like before, one can check that these local tensors patch together to give a global object on Z. We also still have that f is projective if and only if $S(f) = 0$.

Now let X be a Riemann surface and let $\mathcal{P}(X)$ be the set of all complex projective structures that have underlying complex structure X (up to isotopy). We can use the Schwarzian derivative to measure the 'difference between $Z, W \in \mathcal{P}(X)$. To do this, note that the identity is a map $Id : Z \to W$ and define

$$Z - W = S(Id) \in Q(X).$$

This is actually a good measure of the difference because in charts we have $S(Id) = z^* S(w \circ z^{-1})$. So $S(Id)$ is measuring the projective compatibility between the projective atlas for Z and the projective atlas for W. If we fix a basepoint $Z_0 \in \mathcal{P}(X)$, we can define an isomorphism $\mathcal{P}(X) \to Q(X)$ by sending $Z \mapsto Z - Z_0$. Hence $\mathcal{P}(X)$ is an affine space modeled on the vector space $Q(X)$.

The viewpoint taken here is due to Thurston (see [Thu86]). See also [And98] and [Dum09] for further discussion.

References

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